C. Spin Magnetic Moment associated with Spin angular Momentum

- Very rough (don't take it seriously) picture
  - electron (charge $-e$) spinning $\Rightarrow$ Magnetic Moment due to spin AM
- From experiments (Stern-Gerlach type for example)

$$\boxed{\vec{\mu}_S = -\frac{e}{m_e}\vec{S}} \quad (8) \qquad \text{c.f. } \vec{\mu}_L = \frac{-e}{2m_e}\vec{L}$$

Magnetic Moment due to electron's spin AM

charge $(-e)$

electron's spin AM

(orbital AM)

- Generally, for a particle of spin $\vec{S}$, charge q, and mass m, write

$$\vec{\mu}_S = g_S \frac{q}{2m}\vec{S}$$

with g_S (often simply called the g-factor "g") determined experimentally

- For electron, g is known to great accuracy! ($q = -e$, $m = m_e$)

  $g(\text{electron}) = 2.002\,319\,304\,361\,82$ (NIST, USA)

  $\therefore\ g_s = 2$ is a good approximation

  Thus, $\boxed{\vec{\mu}_s = -\frac{e}{m_e}\vec{S} \quad \text{for electron spin}}$ (Take-home message)

- QED (Quantum Electrodynamics) gives highly accurate calculations of $g(\text{electron})$ (the most accurate theory ever!)
  - QED is a Quantum Field Theory (QFT)

<u>Recall</u>: Spin Angular Momentum of electron (an intrinsic[†] property)

S_z can take on only $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$

$|\vec{S}|$ can take on $\sqrt{\frac{3}{4}}\,\hbar$ (a constant) only

$\therefore\ |\vec{S}| = \sqrt{s(s+1)}\,\hbar$ with $\underbrace{s = 1/2}_{\text{always } 1/2 \text{ for an electron}}$ "spin-half"

$S_z = m_s \hbar$ where $m_s = \underbrace{+1/2,\ -1/2}_{\text{2 values only}}$

[†] To be contrast with $\vec{\mu}_L \propto -\vec{L}$

depending on "ℓ" of the electron (motion of electron) & ℓ takes on $0, 1, \cdots$ (not a single value) (not an intrinsic property)

- Back to $\vec{\mu}_s = -\frac{e}{m_e}\vec{S}$, $\quad |\vec{\mu}_s| = \frac{e}{m_e}|\vec{S}| = \frac{e}{m_e}\hbar\sqrt{\frac{3}{4}} = \sqrt{3}\,\frac{e\hbar}{2m_e} = \sqrt{3}\,\mu_B$

$\therefore$ $\underbrace{|\vec{\mu}_s|}$ $= \sqrt{3}\,\mu_B$ (<u>always</u>, one value† for all electrons' spin AM) (9)

$\boxed{\text{Magnetism in Solids come from electron's spin}}$

- $\mu_{s,z} = \frac{-e}{m_e} S_z$

$$\boxed{\begin{array}{lll} & \mu_{s,z} \text{ takes on} & \\ \text{either} & \frac{-e}{m_e}\cdot\left(\frac{\hbar}{2}\right) & \text{OR} \quad \frac{-e}{m_e}\left(-\frac{\hbar}{2}\right) \\ & = -\mu_B & \quad\quad = +\mu_B \\ & (\text{for } m_s = +\tfrac{1}{2}) & \quad\quad (\text{for } m_s = -\tfrac{1}{2}) \end{array}} \quad (10)$$

- Expect $\vec{\mu}_s$ (like $\vec{\mu}_L$) to have some effects in the presence of $\vec{B}$
[related to Zeeman Effect? see later]

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† This point, looks trivial, is often ignored. But it is this point that makes some materials (e.g. Co, Fe, Ni) magnetic.

Take-Home Messages

- Electron has an intrinsic spin angular momentum
- Accompanying $\vec{S}$ is $\vec{\mu}_s = -\frac{e}{m_e}\vec{S} = -\underbrace{g_s}_{\text{"2"}}\frac{e\hbar}{2m_e}\frac{\vec{S}}{\hbar} = -g_s\mu_B\frac{\vec{S}}{\hbar}$

$\therefore$ $\boxed{\text{Electron has an intrinsic spin magnetic dipole moment}}$

$|S| = \sqrt{\frac{3}{4}}\,\hbar$ (always) $\Rightarrow$ $|\vec{\mu}_s| = \sqrt{3}\,\mu_B$ (always for spin-half particle)

$S_z \nearrow \hbar/2$, $\searrow -\hbar/2$ $\Rightarrow$ $\mu_{s,z} \nearrow -\mu_B$, $\searrow +\mu_B$ $\Big\}$ two values only

$\therefore$ $\boxed{\mu_B \text{ sets the strength of spin magnetic dipole moment of an electron and thus atom}}$

Atomic states including spin

- Schrödinger QM only gives $\psi_{n l m_l}(r,\theta,\phi)$ [spatial part] and cannot give the electron's spin [spin part of wavefunction] naturally
- In Schrödinger QM, spin part is included by hand

e.g. without spin H-atom 1s $\psi_{100}(\vec{r}) = \psi_{100}(r,\theta,\phi)$

with spin: H-atom 1s spin-up state $\psi_{100}(r,\theta,\phi)\cdot\alpha_z \equiv \psi_{100,+1/2}$

spin-up state

$n\ l\ m_l,\ m_s$

($s=1/2$ always)

H-atom 1s spin-down state $\psi_{100}(r,\theta,\phi)\cdot\beta_z \equiv \psi_{100,-1/2}$

$m_s=-1/2$

spin-down state

α_z, β_z are eigenstates of $\hat{S}_z$

<u>High-school</u>

- (Some) atoms behave as tiny magnets

<u>University</u>

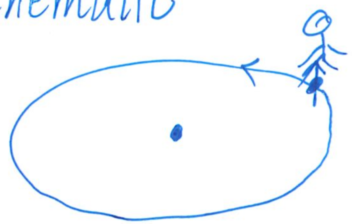
- (Some) atoms behave as tiny magnets due to the effect of the spin magnetic dipole moments of the electrons in an atom (typically of order μ_B per atom)

Consider $2p_z$ spin-up state $\quad R_{21}(r)\,Y_{10}(\theta,\phi)\cdot\alpha_z$

$\psi_{210}(\vec{r})\cdot\alpha_z$

Schematic

electron

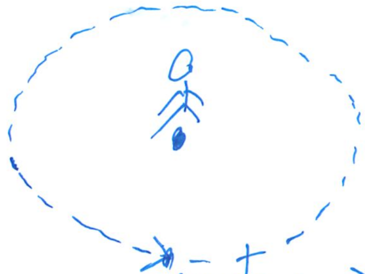


- $l = 1 \Rightarrow$ has $|\vec{L}| = \sqrt{2}\,\hbar$ (orbital AM)
- $m_s = +1/2$

  has $|\vec{S}| = \sqrt{\frac{3}{4}}\,\hbar$ (spin AM)

$$\boxed{\text{Is there a total AM } \vec{J} = \vec{S} + \vec{L}\ ?}$$

relative motion (due to $\vec{L}$)



nucleus $\Rightarrow \vec{B}_{int}$ at electron ($\vec{\mu}_s$)

$$\boxed{\text{Will } \vec{\mu}_s\ (\propto -\vec{S}) \text{ couple with own } \vec{L}\ ?}$$

Appendix: Quick Review on Spin Angular Momentum of Electron‡

- $\underbrace{\text{Intrinsic}}$ property of electron

  Meaning: Same property for all electrons (anywhere, everywhere)

- S^2 of an electron† is $\left(\frac{3}{4}\hbar^2\right)$ [a constant, true for all electrons]

  (c.f. all electrons have $\underbrace{\text{charge}}_{\text{another intrinsic property}}$ $(-e)$)

- $\frac{3}{4}\hbar^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2$

  $= s(s+1)\hbar^2$

$\therefore$ $\boxed{\text{Electron's spin angular momentum is characterized by } s = 1/2}$ (A1)

"Electron is a spin-half particle"

$|S| = \sqrt{\frac{3}{4}}\,\hbar$ for all electrons (Magnitude of electron's spin AM)

† Recall: Stern-Gerlach experiment

‡ This is included here for completeness. See earlier class notes for details.

- $S = 1/2 \Rightarrow$ any component (usually $\hat{S}_z$) can take on

$$\text{either} \quad \underbrace{+\frac{\hbar}{2}}_{m_s = +1/2} \quad \text{OR} \quad \underbrace{-\frac{\hbar}{2}}_{m_s = -1/2} \quad \text{(2 values only)} \qquad (A2)$$

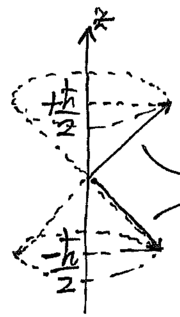
$\underbrace{|s, m_s\rangle}$ (c.f. $|j, m_j\rangle$ for general[+] angular momentum in QM)

$$|s=1/2,\ m_s=+1/2\rangle = |1/2, 1/2\rangle \quad \text{"spin-up" OR } \alpha_z$$
$$|s=1/2,\ m_s=-1/2\rangle = |1/2, -1/2\rangle \quad \text{"spin-down" OR } \beta_z \qquad (A3)$$

Since $s = 1/2$ (always) for all electrons, a short-hand notation is to specify m_s only, i.e. $|m_s = +1/2\rangle$ OR $|m_s = -1/2\rangle$

[+] Recall: Given j, m_j runs from $-j$ to $+j$ in steps of 1
[J_z can take on $m_j\hbar$ with $m_j = -j, -j+1, \cdots, j-1, j$]

Vector Model (just for illustrating key features, not to be taken seriously)



length $= \sqrt{\frac{3}{4}}\,\hbar$

- Projection on $\hat{z}$-direction is either $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$
- When S_z is certain, S_y and S_x are uncertain.

Mathematical Structure of spin-half particle

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x \; ; \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y \; ; \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_z \qquad (14)$$

Hence, $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma_x, \sigma_y, \sigma_z$: Pauli Matrices

They satisfy the commutation relations.

α_z (spin-up of $\hat{S}_z$) $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

β_z (spin-down of $\hat{S}_z$) $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Appendix: Angular Momentum in Quantum Mechanics: A quick review[†]

- A quantity (general symbol) $\vec{J}$ with components J_x, J_y, J_z; and magnitude squared J^2, for which their operators satisfying the commutation relations

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z, \quad [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x, \quad [\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y$$

(note cyclic pattern) (A4)

$$[\hat{J}^2, \hat{J}_x] = 0, \quad [\hat{J}^2, \hat{J}_y] = 0, \quad [\hat{J}^2, \hat{J}_z] = 0$$

is an Angular Momentum in Quantum Mechanics

- AM is defined by the commutation relations (A4) in QM
- Can find simultaneous eigenstates of $\hat{J}^2$ and one component (say $\hat{J}_z$)

one label: j another label: m_j

† See chapters on orbital, general, and spin angular momentum for details.

- It follows from (44) alone[+] that the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$ must take on certain values

Let $|j, m_j\rangle$ be simultaneous eigenstates of $\hat{J}^2$ and $\hat{J}_z$

$$\hat{J}^2 |j, m_j\rangle = j(j+1)\hbar^2 |j, m_j\rangle$$

with j being integers $(0, 1, 2, \cdots)$ OR half-integers $(1/2, 3/2, 5/2, \cdots)$ (A5)

$$\hat{J}_z |j, m_j\rangle = m_j \hbar |j, m_j\rangle$$

with $m_j = \underbrace{j, j-1, \cdots, -j+1, -j}_{(2j+1)\text{ values}}$ for a given value of j

- Why bother with <u>General</u> AM?
  - Nature makes use of AM cleverly $\Rightarrow$ <u>Many</u> angular momenta

    e.g. simplest atom (<u>Hydrogen atom</u>)

    electron has orbital AM ($Y_{\ell m_\ell}(\theta,\phi)$)

    electron has spin AM ($|S| = \sqrt{\frac{3}{4}}\hbar$; $S_z = \frac{1}{2}\hbar, -\frac{1}{2}\hbar$)

    electron has total AM ($\vec{J} = \vec{L} + \vec{S}$) ($\vec{J}$ is an angular momentum)

    proton (nucleus) has spin AM

    e.g. Many-electron atoms (all other atoms)

    need to consider $\underbrace{\text{adding up spin AM}}_{\text{still an AM}}$ of electrons

    $\overbrace{\text{adding up orbital AM}}^{\text{still an AM}}$ of electrons

- All angular momenta obey (A4) and (A5).